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Multidimensional stability analysis of gaseous detonations near Chapman–Jouguet conditions for small heat release

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Multidimensional instability of planar detonations, leading to cellular structures, is studied analytically near Chapman–Jouguet conditions, in the limit of small heat release, with small (Newtonian) differences between heat capacities, by using an expansion in a small parameter representing the ratio of the heat release to the thermal enthalpy of the fresh mixture. In this limit, the dynamics of detonations is governed by the interaction between the acoustic waves and the heat-release rate inside the inner detonation structure, the entropy–vorticity wave playing a negligible role at leading order. This situation is just opposite from that considered in our 1997 study of strongly overdriven detonations. The present analysis offers a step towards improving our understanding of the cellular structures of ordinary detonations, for which both the entropy–vorticity waves and the acoustic waves are involved in the instability mechanism. The relevant bifurcation parameter is identified, involving the degree of overdrive and the sensitivity of the rate of heat release to temperature at the Neumann state, and the onset of the instability is studied analytically for a realistic model of the inner structure of gaseous detonations.

1. Introduction

The Zeldovich–Neumann–Döring structure of gaseous detonation is a strong inert shock followed by a reaction zone. This zone is controlled by the complex chemical kinetics of the combustible mixtures. In general, two distinct layers are identified, an induction zone (adjacent to the shock) followed by an exothermic zone. Across the induction zone, the rate of heat release increases very slowly from zero, and, therefore, the temperature does not increase much. At the end of this induction zone, a transition occurs, the rate of heat release increases strongly and decreases afterwards continuously, up to the end of the exothermic layer. The lengths of these two zones are of a similar order of magnitude and both are much larger than the shock thickness which is, therefore, considered as hydrodynamic discontinuity across which the Rankine–Hugoniot jump conditions are satisfied. The gas flow is subsonic relative to the shock, but its velocity is sufficiently large so that the heat conduction and the molecular diffusion can be neglected.

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The dynamics of gaseous detonations is thus obtained by solving the Euler equations with exothermic reactions and with boundary conditions at a free boundary (the perturbed shock). This is a challenging problem, even in the linear approximation which is sufficient for addressing the stability of the unperturbed planar wave. The linear modes of an ideal gas flow are easily obtained when the unperturbed solution is uniform. They take the well-known form of acoustic waves plus an isobaric wave, called an entropy–vorticity wave, propagating at the velocity of the unperturbed flow. These modes are more difficult to identify when the density and temperature of the unperturbed medium vary in space, as is the case across the detonation structure. The difficulty is further increased for self-sustained detonations, called Chapman–Jouguet detonations, by a transonic condition appearing at the end of the exothermic reaction. This difficulty no longer exists for overdriven regimes of piston-supported detonations across which the flow of shocked gas is subsonic everywhere.

The feedback loop controlling the detonation dynamics may be described schematically as follows. Disturbances of the shock velocity perturb the rate of heat release in the bulk gas, essentially by modifying the induction length, which is sensitive to temperature variations. Those perturbations propagate downstream with two modes at two different velocities (sound speed and flow velocity), a downstream-running acoustic mode and an entropy–vorticity wave. The modifications of heat release, in turn, perturb the flow and thereby affect the shock velocity, through the Rankine–Hugoniot conditions, after a delay associated with an upstream-running acoustic wave.

This hyperbolic problem may be solved numerically, either by direct numerical simulations as explained in the book of Oran & Boris (1987) (see also Bourlioux & Majda 1992), or by solving the linear stability equations (written in a Fourier-mode decomposition) by a shooting algorithm, as Short & Stewart (1998) did. The first results concerning the stability analysis were obtained in the 1960s by the pioneering work of Erpenbeck within the framework of the standard idealized model of an irreversible unimolecular reaction with an Arrhenius rate in a series of papers starting with Erpenbeck (1962) and ending with Erpenbeck (1966). It was Erpenbeck who apparently first realized that detonations become unstable already when the heat release is small compared with the thermal enthalpy (see Erpenbeck 1964, 1965) The stability problem is sufficiently complicated that understanding is best advanced by exercising a number of different analytical approaches. Some of those carried out more recently, initiated by Buckmaster (1989), include asymptotic analyses for large activation energy, assuming simplified scalings of time and length (see Short & Stewart 1998 for an extensive review of the literature). Some others are based only on asymptotic expansions in different parameters, for a large Mach number of propagation considered by Clavin & He (1996*b*) and Clavin, He & Williams (1997), or for small heat release studied by Short & Stewart (1999), the scaling of time resulting from the analysis.

By combining a large propagation Mach number with the Newtonian approximation (which treats a small difference between the specific heats at constant pressure and at constant volume, $(\gamma - 1) \ll 1$), Clavin & He (1996*b*) in one-dimensional geometry and Clavin *et al.* (1997) for the multidimensional stability consider the case of strongly overdriven detonations for a general chemistry of combustion. In this limit, the shocked flow is sufficiently subsonic so that the quasi-isobaric approximation of low-Mach-number flow is satisfied everywhere downstream from the shock. The only mode that controls the dynamics is then the entropy–vorticity wave. For a given wavenumber of the disturbance, the linear growth rate is obtained as a solution of an integral equation involving a single delay and four non-dimensional quantities: two functions of space $\overline{w}(\xi)$ and $\overline{w}'(\xi)$, $\int_0^{+\infty} \overline{w}(\xi)\,\mathrm{d}\xi = 1$ and $\int_0^{+\infty} \overline{w}'(\xi)\,\mathrm{d}\xi = 0$, plus two scalar parameters representing respectively the large propagation Mach number (the overdrive factor) and the total amount of chemical heat release. The functions $\overline{w}(\xi)$ and $\overline{w}'(\xi)$ are, respectively, the reduced spatial distribution of the heat-release rate of the planar wave and its deformation when varying the overdrive factor, with ξ being the reduced distance from the leading shock, normalized, for example, by the unperturbed induction length. The transition from stable to unstable detonations is then obtained, showing how detonations that are stable against planar disturbances may become unstable against multidimensional disturbances. The instability is found to be promoted by an increase in the thermal sensitivity of the heat-release rate, or by a decrease of the overdrive factor. These results also show that the frequencies of the weakly unstable linear modes, which appear at the bifurcation, increase when increasing the stiffness of the functions $\overline{w}(\xi)$ and $\overline{w}'(\xi)$, leading to pathological dynamics for singular distributions of heat release, as for example in the limit of an infinite activation energy of the standard model, yielding the so-called square-wave model, $\overline{w}(\xi) = \delta(\xi - 1)$, $\overline{w}'(\xi) = -h\delta'(\xi - 1)$, with h being a positive parameter measuring the sensitivity of the induction length to temperature, and $\delta(\xi)$ and $\delta'(\xi)$ denoting the Dirac delta function and its derivative, respectively. A more surprising result is that the small effects of the acoustic waves, which were included afterwards by a perturbation analysis in the one-dimensional case, promote the stability of the detonation wave (see Clavin & He 1996*a*). Studies of the properties of the instability near the bifurcation point are important because they provide a necessary preliminary step in developing a weakly nonlinear analysis of the dynamics for explaining the experimentally observed diamond patterns of cellular detonations (see the bifurcation analysis of Clavin & Denet 2002 with technical details in Clavin 2002*a*, *b*, or, for a different approach, see Yao & Stewart 1996).

An open question is that of determining whether or not, or to what extent, the conclusions that have been obtained by the multidimensional stability analysis of strongly overdriven detonations, remain valid for detonations close to the Chapman–Jouguet regime, where, obviously, the compressible effects can no longer be neglected at leading order in the flow downstream from the shock. This question cannot be addressed analytically without further approximations. The purpose of the present paper is to approach this question by investigating analytically a limiting case, opposite to the one studied by Clavin *et al.* (1997) and in which the influence of the entropy–vorticity wave is negligible, with the dynamics of the detonation being entirely controlled by the interaction of the acoustic waves and the heat release. This is the case for Chapman–Jouguet detonations with small heat release, the dynamics of which was studied by Clavin & Williams (2002) in one-dimensional geometry. The present paper is an extension of this previous work to the multidimensional stability analysis.

The proximity of the Chapman–Jouguet regime was not specifically considered in the stability analysis of Short & Stewart (1999). Their analysis concerns the standard one-step model for a small ratio of the chemical heat release to the shocked-gas enthalpy at the shock (the Neumann state), without approximation for γ. For ordinary values of γ, their results agree well with those of Clavin *et al.* (1997) at very large overdrive, but they become less accurate when the overdrive factor is decreased, as was shown by a comparison with a numerical analysis (see Daou & Clavin 2003). The smallness of their parameter implies large overdrive and precludes approach to Chapma–Jouguet conditions. This situation is easily explained by noticing that, for a very large overdrive factor, the increase in temperature across the leading shock becomes larger than that associated with the heat release downstream from the shock, while, in the Newtonian limit, the quasi-isobaric approximation is still valid for the smaller overdrive factors for which these two increases in temperature are of the same order.

The detonation structure that is considered here, namely the structures for regimes near Chapmpan–Jouguet conditions for small ratios of the chemical heat release to the enthalpy of the fresh gas mixture, is of a quite different nature. The conditions in the gas flow are close to a transonic regime everywhere. Such detonations cannot easily be observed in experiments, since quenching of the exothermic reaction occurs at low temperature, according to the complex chemical kinetics of the combustion of fuels such as hydrogen or hydrocarbons. These regimes are, however, worth investigating theoretically because of the light that the results can shed on the role of the acoustic waves in the dynamics of detonations, in particular by comparison with the entropy–vorticity wave exhibited at large overdrive. For ordinary detonations, both types of waves are present in the flow of the shocked gas. Studying them separately is a good strategy for improving our understanding of the dynamics of detonation.

The paper is self-contained. The chemical reacting Euler equations are recalled in § 2. A particular form of these equations, which turns out to be useful in the limit considered later, is also given in this section, where the general conditions at the shock and their linear approximation are also recalled. Particular attention is paid to the conditions at infinity in § 3. The parameters involved in the asymptotic limit are presented in § 4, with the corresponding expansions of the conditions at the shock. The stability of detonations against disturbances with large wavelength, larger than the detonation thickness, is described in § 5. The length and time scalings for unstable disturbances are discussed in § 6, where the reduced equations at leading order are also obtained, together with the corresponding boundary values. Approximations for the heat-release rate are discussed in § 7. Expressions for the oscillatory frequency, the linear growth rate and the wavelength of unstable detonations are obtained in § 8, near bifurcation, from unconditionally stable-to-unstable detonations. The bifurcation conditions expressed in terms of the parameters defining the propagation regimes are also discussed in this section. Conclusions are given in § 9.

2. The basic equations

A two-dimensional time-dependent chemically reacting Euler flow of an ideal gas is considered, with x denoting the coordinate in the main flow direction, y the transverse coordinate and t time. Velocity components in the x and y directions are denoted by u and v, respectively, $\boldsymbol{u} = (u, v)$, and the leading shock of the steady planar detonation whose stability is to be investigated is placed at $x = 0$. The Euler equations are studied downstream from this shock, chemistry presumed not to occur ahead of it. Expressed in terms of the density ρ and temperature T, the sound speed a and pressure are given by

$$a = \sqrt{\gamma \frac{p}{\rho}}, \qquad p = \frac{(\gamma - 1)}{\gamma} c_p \rho T, \tag{2.1}$$

where γ is the ratio of the specific heat at constant pressure c_p to the specific heat at constant volume. The equation for mass, momentum and energy can be written, respectively, as

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \nabla\cdot\boldsymbol{u} = 0, \qquad \rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p, \qquad \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{Q}{c_pT}\frac{w}{t_r}, \tag{2.2}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{u}\cdot\nabla$ is the material derivative, Q denotes the chemical heat release per unit mass of mixture, t_r is a representative chemical reaction time and w is a non-dimensional function of state variables that describes the rate of chemical heat release. It is convenient to let Y denote the fraction of chemical heat that has been released. The complex systems of equations for describing a detailed chemical-kinetic scheme are not addressed explicitly here but instead are written schematically in the form

$$\frac{\mathrm{D}Y}{\mathrm{D}t} = \frac{w(Y,T)}{t_r}, \tag{2.3}$$

where w is regarded as a function of Y and T, the dependence of the reaction term on the pressure being neglected for simplicity, the dependence on temperature usually being much stronger. An alternative and often more useful form of the energy conservation equation, obtainable by use of (2.1) and (2.2), is the entropy equation, which may be written as

$$\frac{1}{\gamma p}\frac{\mathrm{D}p}{\mathrm{D}t} + \nabla\cdot\boldsymbol{u} = \frac{Q}{c_pT}\frac{w}{t_r}, \tag{2.4}$$

where the equation for mass conservation has been used to eliminate the density.

2.1. *A useful form*

For the purpose of stability analysis, second-order nonlinear terms that involve $v\partial/\partial y$ can be neglected, and it is helpful to define the differential operators

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \qquad \frac{\mathrm{D}^{\pm}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}, \tag{2.5}$$

which would lead to the characteristic equation for the one-dimensional problem. The conservation equations to be used for the stability analysis may then be written with the notation (2.5) as

$$\frac{\mathrm{D}}{\mathrm{D}t}Y = \frac{w(Y,T)}{t_r}, \qquad \frac{\mathrm{D}}{\mathrm{D}t}\left[\ln T - \frac{(\gamma-1)}{\gamma}\ln p\right] = \frac{Q}{c_pT}\frac{w}{t_r}, \tag{2.6}$$

$$\frac{1}{\gamma}\frac{\mathrm{D}^{\pm}}{\mathrm{D}t}\ln p \pm \frac{1}{a}\frac{\mathrm{D}^{\pm}}{\mathrm{D}t}u = \frac{Q}{c_pT}\frac{w}{t_r} - \frac{\partial}{\partial y}v, \qquad \frac{\mathrm{D}}{\mathrm{D}t}v = -\frac{1}{\bar{\rho}}\frac{\partial}{\partial y}p, \tag{2.7}$$

where the overbar identifies the steady planar solution whose stability is investigated. The first equation in (2.7) resembles the characteristic equations of the one-dimensional problem. It is obtained when the equation for conservation of the longitudinal momentum in (2.2) is multiplied by $a/(\gamma p) = 1/(a\rho)$ and added to and subtracted from (2.4). Equations (2.7) are useful for studying the perturbations of the flow across the perturbed detonation structure when both the heat release and the transverse component of the flow velocity are small (see § 6).

These differential equations are subject to boundary conditions at the shock, where $x = 0$ to leading order in the linear approximation and at $x \to +\infty$. In this paper, we shall let the subscript u represent conditions ahead of the shock and b conditions as $x \to +\infty$. The condition in the compressed gas at the shock (the Neumann state) will be represented by the subscript N.

2.2. *Conditions at the shock*

Let $x = A(y, t)$ represent the perturbed shock position at transverse position y at time t. The normal component to the shock of the relative velocity with respect to the shock of the upstream flow is then $(\overline{u}_u - \partial A/\partial t)/\sqrt{1 + (\partial A/\partial y)^2}$. Since the linear analysis implies neglecting terms quadratic in $(\partial A/\partial y)$, the corresponding Mach number is $M_U = M_u - (\partial A/\partial t)/\overline{a}_u$, where $\overline{a}_u$ and M_u are the sound speed ahead of the shock and the propagation Mach number of the planar detonation. M_U serves as the approach-flow Mach number for writing shock jump conditions, which then turn out to be the same as those for the one-dimensional problem,

$$\frac{p_N}{\overline{p}_u} = 1 + \frac{2\gamma}{\gamma+1}\left(M_U^2 - 1\right), \quad \frac{\rho_N}{\overline{\rho}_u} = \frac{1 + \left(M_U^2 - 1\right)}{1 + ((\gamma-1)/(\gamma+1))\left(M_U^2 - 1\right)}. \tag{2.8}$$

This can be employed to derive the perturbations of pressure and density at the Neumann state

$$\frac{\delta p_N}{\gamma \overline{p}_N} = -\frac{4}{(\gamma+1)}\frac{M_u}{\left[1 + (2\gamma/(\gamma+1))\left(M_u^2 - 1\right)\right]}\frac{\partial A/\partial t}{\overline{a}_u}, \tag{2.9}$$

$$\frac{\delta \rho_N}{\overline{\rho}_N} = -\frac{4}{(\gamma+1)M_u}\frac{1}{\left[1 + ((\gamma-1)/(\gamma+1))\left(M_u^2 - 1\right)\right]}\frac{\partial A/\partial t}{\overline{a}_u}. \tag{2.10}$$

The temperature at the front is obtained from these equations when using the perfect gas law (2.1). The additional requirement of continuity of the tangential component of velocity $\delta v_N = (\overline{u}_u - \overline{u}_N)\partial A/\partial y$, and the mass conservation $\rho_N(u_N - \partial A/\partial t) = \overline{\rho}_u(\overline{u}_u - \partial A/\partial t)$, namely $\delta u_N = (1 - \overline{\rho}_u/\overline{\rho}_N)\partial A/\partial t - \overline{u}_N(\delta\rho_N/\overline{\rho}_N)$, can be employed to derive

$$\frac{\delta u_N}{\overline{a}_u} = \frac{2\left(M_u^2 + 1\right)}{(\gamma+1)M_u^2}\frac{\partial A/\partial t}{\overline{a}_u}, \qquad \frac{\delta v_N}{\overline{a}_u} = \frac{2\left(M_u^2 - 1\right)}{(\gamma+1)M_u}\partial A/\partial y, \tag{2.11}$$

which shows that δu_N is of order $\partial A/\partial t$ irrespective of whether M_u is near unity or large, while δv_N is small compared with $\overline{a}_u \partial A/\partial y$ when M_u is near unity, but large compared with this quantity when M_u is large.

3. Conditions at infinity

3.1. *General considerations*

The steady problem can be viewed as one in which the detonation is supported by a piston moving at the constant subsonic velocity $\overline{u}_b$, $M_b = \overline{u}_b/\overline{a}_b \leqslant 1$, $\overline{a}_b$ denoting the sound speed in the burnt gases. The y-axis is along the unperturbed detonation front, and the unperturbed flow is in the direction of positive x, $\lim_{x\to+\infty}\overline{u} = \overline{u}_b > 0$, $\lim_{x\to-\infty}\overline{u} = \overline{u}_u > 0$, $M_u = \overline{u}_u/\overline{a}_u > 1$. We shall not address complications associated with reversible chemical reactions possibly occurring in these products, but instead discuss intricacies in downstream boundary conditions for the detonation imposed by the presence of an inert ideal gas constituting a uniform medium downstream. The stability analysis of the detonation requires proper selections to be made in the perturbed solutions in the downstream gas, expressed here for any function $f(x, y, t)$ in the Fourier representation $\delta f(x)e^{iky+\sigma t}$, with k being the real parts representing physical variables.

Attention must be given to the fact that, according to the linear solution of the conservation equations in a uniform medium, the downstream gas can support both acoustic waves and entropy–vorticity waves. The pressure disturbance obeys the d'Alembert equation with sound speed $\overline{a}_b$ and thus must have $\delta p(x) = \delta p_b e^{\mathrm{i}l_\pm x}$ as the form of the perturbation solution, where δp_b is a constant, and

$$\frac{\mathrm{i}l_\pm}{|k|} = \frac{M_b(\sigma/\overline{a}_b|k|) \pm \sqrt{\left(1-M_b^2\right) + (\sigma/\overline{a}_b k)^2}}{\left(1-M_b^2\right)}, \qquad \delta p = \delta p_b e^{\sigma t + \mathrm{i}(l_\pm x + ky)}, \tag{3.1}$$

which approaches the well-known result $\mathrm{i}l_\pm = \pm k$ in the limit of infinite sound speed (zero Mach number). The quantity $\sigma/\overline{a}_b k$ may be expressed in terms of $l_\pm$ and k from (3.1), leading to the well-known result for the frequency of a purely oscillatory acoustic mode, $\sigma = \mathrm{i}\omega$, $\omega > 0$, $\omega^2 > (1 - M_b^2)\overline{a}_b^2 k^2$,

$$\omega = \overline{a}_b \sqrt{l_\pm^2 + k^2} - \overline{u}_b l_\pm, \tag{3.2}$$

which shows the frequency shift of the Doppler effect. The solutions for the perturbed velocity components are the sum of an acoustic wave and an entropy–vorticity wave which is simply convected by the unperturbed flow velocity $\delta \boldsymbol{u}_{ev}(x - \overline{u}_b t, y)$,

$$\delta u(x) = -\frac{\mathrm{i}l_\pm \overline{u}_b}{\sigma + \mathrm{i}l_\pm \overline{u}_b}\frac{\delta p_b}{\overline{\rho}_b \overline{u}_b} e^{\mathrm{i}l_\pm x} + \left[\delta u_b + \frac{\mathrm{i}l_\pm \overline{u}_b}{\sigma + \mathrm{i}l_\pm \overline{u}_b}\frac{\delta p_b}{\overline{\rho}_b \overline{u}_b}\right] e^{\sigma \frac{x}{\overline{u}_b}}, \tag{3.3}$$

$$\delta v(x) = -\frac{\mathrm{i}k \overline{u}_b}{\sigma + \mathrm{i}l_\pm \overline{u}_b}\frac{\delta p_b}{\overline{\rho}_b \overline{u}_b} e^{\mathrm{i}l_\pm x} + \left[\delta v_b + \frac{\mathrm{i}k \overline{u}_b}{\sigma + \mathrm{i}l_\pm \overline{u}_b}\frac{\delta p_b}{\overline{\rho}_b \overline{u}_b}\right] e^{\sigma \frac{x}{\overline{u}_b}}, \tag{3.4}$$

with δu_b and δv_b also being constants. Here each first term is an acoustic mode, like the solution $\delta p(x)$, the rest being the vorticity wave. This last wave must satisfy the incompressibility condition, $\partial \delta u_{ev}/\partial x + \partial \delta v_{ev}/\partial y = 0$, found from (3.3) and (3.4) to obtain

$$\pm\sqrt{\left(1-M_b^2\right)k^2 + \left(\frac{\sigma}{\overline{a}_b}\right)^2}\,\frac{\delta p_b}{\overline{\rho}_b \overline{u}_b} + \sigma\frac{\delta u_b}{\overline{u}_b} - \mathrm{i}k\delta v_b = 0, \tag{3.5}$$

with the same sign convention as in (3.1). Equation (3.5), which was first written explicitly by Buckmaster & Ludford (1988), is obtained by using the relation

$$\mathrm{i}l_\pm \sigma + \overline{u}_b k^2 = \pm(\sigma + \mathrm{i}l_\pm \overline{u}_b)\sqrt{\left(1-M_b^2\right)k^2 + (\sigma/\overline{a}_b)^2},$$

when the squared root is expressed according to (3.1). With the perfect gas law $\gamma p = \rho a^2$, (3.5) takes the form

$$\pm\sqrt{\left(1-M_b^2\right) + \left(\frac{\sigma}{\overline{a}_b k}\right)^2}\,\frac{\delta p_b}{\gamma \overline{p}_b} = -\left(\frac{\sigma}{\overline{a}_b k}\right)\frac{\delta u_b}{\overline{a}_b} + \mathrm{i}M_b \frac{\delta v_b}{\overline{a}_b}. \tag{3.6}$$

3.2. *Planar case*

The planar limit of result (3.5), $k \to 0$, has $\delta u_b = \pm\, \delta p_b/\overline{\rho}_b \overline{a}_b$ and reproduces from (3.1) the classical one-dimensional acoustic dispersion relation in a moving medium $\sigma = \mathrm{i}l\overline{a}_b(1 \mp M_b)$, exhibiting the Doppler effect. Since δp is purely acoustic, this shows that any streamwise velocity perturbation is excluded from occurring in the entropy wave in the planar limit, this wave carrying only temperature and density fluctuations.

3.3. *Boundedness condition*

Unstable acoustical modes, namely those having $\mathrm{Re}(\sigma) > 0$, are physically relevant only if they are bounded at infinity, $\mathrm{Re}(\mathrm{i}l_\pm) \leqslant 0$, $l_\pm^{(i)} > 0$, $l_\pm = l_\pm^{(r)} + \mathrm{i}l_\pm^{(i)}$. Adopting the normal convention that real parts of square roots are taken to be positive, we find that the negative sign must be selected in (3.1), $l = l_-$, and thus also in (3.5) and (3.6). This removes one of the two acoustic eigenmodes and enables a dispersion relation to be obtained from (3.5), which then serves as a compatibility condition for the perturbations (δp_b, δu_b, δv_b) at the exit of the reaction zone (the entrance to the inert region downstream). In the planar limit this reduces simply to $\delta u_b = \delta p_b / \overline{\rho}_b \overline{a}_b$, the downstream boundary condition for the reaction zone employed previously for the planar problem.

In multidimensional problems, the sign restriction corresponds to a radiation condition that requires acoustic waves associated with unstable modes to propagate downstream, preventing such waves from travelling upstream from infinity to impinge on the detonation,

$$\text{radiation condition:} \qquad \overline{u}_b - \overline{a}_b \frac{l_-^{(r)}}{\sqrt{l_-^{(r)2} + k^2}} > 0, \tag{3.7}$$

where the second term is the x-component of the propagation velocity of the acoustic wave in the moving frame of the burnt gas. The radiation condition, $M_b^2 k^2 > (1 - M_b^2) l_-^{(r)2}$, is easily proved from (3.1) for unstable conditions close to a Hopf bifurcation, $\sigma = \mathrm{i}\omega + s$, with $0 < s \ll \omega$ and $\omega^2 < (1 - M_b^2)\overline{a}_b^2 k^2$.

3.4. *Neutral modes*

When the stability analysis exhibits neutral modes only $\mathrm{Re}(\sigma) = 0$, one has usually $l^{(i)} = 0$, $l = l^{(r)}$, and a different interpretation is needed (see D'Yakov 1954 and Kontorovich 1957). Effects of incoming sound waves from infinity and their reflection when impinging the shock then need to be considered. When there is an eigenmode satisfying a radiation condition and the reflected wave matches the radiating eigenmode, the reflection coefficient diverges, implying instability. Shocks and detonations in ideal gases, with internal structure neglected, have neutral modes only, but no radiating modes,

$$\text{incoming sound wave (upstream-running mode):} \quad \overline{u}_b - \overline{a}_b \frac{l_\pm}{\sqrt{l_\pm^2 + k^2}} < 0, \tag{3.8}$$

and therefore are stable according to this criterion (see § 5 for the case of detonations). Instabilities, however, arise when the internal detonation structure is taken into account.

4. Near CJ conditions for small heat release

We are interested in the limit of small heat release, and, as in the study of the one-dimensional instability of Clavin & Williams (2002) the small expansion parameter ε is defined as

$$\varepsilon = \left[\left(\frac{\gamma + 1}{2} \right) \frac{Q}{c_p T_u} \right]^{1/2}. \tag{4.1}$$

According to the Rankine–Hugoniot relations, the Chapman–Jouguet condition corresponds to $(M_u - M_u^{-1}) = 2\varepsilon$, namely $M_u = \sqrt{1 + \varepsilon^2} + \varepsilon$, and planar detonations propagating at constant velocity exist only if $f \geqslant 1$, where

$$f = \left(\frac{M_u^2 - 1}{2\varepsilon M_u} \right)^2 \tag{4.2}$$

is a scaled overdrive factor which decreases to unity at Chapman–Jouguet conditions (see Appendix A). Values of f larger than unity correspond to moderately overdriven regimes, still close to Chapman–Jouguet conditions, provided that the condition $\varepsilon^2 f \ll 1$ is satisfied. To leading order in the limit $\varepsilon\sqrt{f} \to 0$, one gets

$$M_u^2 - 1 \approx 1 - M_N^2 = 2\varepsilon\sqrt{f} + \cdots, \qquad \frac{\overline{T}_N}{\overline{T}_u} - 1 = 2(\gamma - 1)\varepsilon\sqrt{f} + \cdots, \tag{4.3}$$

and from (2.8)–(2.11),

$$\overline{\pi}_N = \frac{4\gamma\varepsilon\sqrt{f}}{\gamma+1} + \cdots, \quad \overline{\theta}_N = 2(\gamma-1)\varepsilon\sqrt{f} + \cdots, \quad \overline{\mu}_N = 1 - \varepsilon\sqrt{f} + \cdots, \tag{4.4}$$

$$\delta\pi_N = -\frac{4}{\gamma+1}\left[1 - \frac{3\gamma-1}{\gamma+1}\varepsilon\sqrt{f} + \cdots\right]\frac{\partial A/\partial t}{\overline{a}_u}, \tag{4.5}$$

$$\delta\theta_N = -2(\gamma-1)(1 + \varepsilon\sqrt{f} + \cdots)\frac{\partial A/\partial t}{\overline{a}_u}, \tag{4.6}$$

$$\delta\mu_N = \frac{4}{\gamma+1}\left[1 - \varepsilon\sqrt{f} + \cdots\right]\frac{\partial A/\partial t}{\overline{a}_u}, \quad \delta\nu_N = \frac{4}{\gamma+1}\varepsilon\sqrt{f}(1+\cdots)\frac{\partial A}{\partial Y}, \tag{4.7}$$

where non-dimensional dependent variables have been introduced,

$$\pi = (1/\gamma)\ln(p/\overline{p}_u), \quad \theta = (T - \overline{T}_u)/\overline{T}_u, \quad \mu = u/\overline{a}_u, \quad \nu = v/\overline{a}_u. \tag{4.8}$$

From (4.3), it is clear that

$$M_u = 1 + \varepsilon\sqrt{f} + \varepsilon^2 f/2 + \cdots$$

which is unity at leading order, emphasizing the transonic character of the problem.

In the limit $\varepsilon \to 0$, the final temperature exceeds the Neumann temperature only if

$$(\gamma - 1) < \frac{\varepsilon}{\sqrt{f} - \sqrt{f-1}}. \tag{4.9}$$

In order to preserve realistic temperature profiles, a Newtonian limit is addressed in which $(\gamma - 1)$ is of order ε or smaller,

$$(\gamma - 1) = O(\varepsilon), \tag{4.10}$$

and the temperature increase of the compressed gas is of order ε^2,

$$\frac{\overline{T}_b - \overline{T}_N}{\overline{T}_u} = -(\gamma-1)\varepsilon(\sqrt{f} - \sqrt{f-1}) + \varepsilon^2 + \cdots, \tag{4.11}$$

where the second term is the chemical heat release and the first term describes a cooling resulting from compressible effects. The conditions at the burnt-gas state are given by (see Appendix A)

$$1 - M_b^2 = 2\varepsilon\sqrt{f-1} + \cdots, \quad \overline{\pi}_b = \varepsilon(\sqrt{f} + \sqrt{f-1}), \quad \overline{\mu}_b = 1 - \varepsilon\sqrt{f-1} + \cdots. \tag{4.12}$$

Note that the relative variations of pressure and velocity across the compressed gas are of order ε, while the relative variation of temperature is smaller, of order ε^2.

5. Stability at long wavelength

Considering detonations that are stable against planar disturbances, modifications to the internal detonation structure can be neglected whenever the wavelength of the disturbances is much larger than the detonation thickness. In such cases, the detonation is approximated by hydrodynamic discontinuity. The dispersion relation is, therefore, obtained by requiring that condition (3.6) is satisfied by the values of pressure and velocities at the burnt-gas side of the detonation; these values being easily obtained from the jump conditions in the planar approximation (see Appendix B). By introducing (A 8), (B 4), (B 10) and (B 11) into (3.6), the result

$$\pm \mathbb{P} S\sqrt{2\epsilon(1+\epsilon\mathbb{M})\sqrt{f-1}+S^2} = -S^2(1-\epsilon\sqrt{f}) - \epsilon\sqrt{f-1}(1+\epsilon\mathbb{V}) \tag{5.1}$$

is obtained, where the notations

$$S = \frac{\sigma}{\overline{a}_b|k|} \quad \text{and} \quad \epsilon = \frac{\varepsilon}{M_u}$$

have been introduced. Observe that the quantities $\pm\sqrt{\cdots}$ in (3.1) and (5.1) are the same except for a factor k^2.

The quadratic equation for S^2 obtained from (5.1) has two negative solutions, so that the eigenmodes correspond to two oscillatory acoustic modes. This can be seen when expression (B 6) for $\mathbb{P}$ is introduced into (5.1) by noticing that the terms of lower order than ϵ^2 in the coefficient in front of both S^4 and S^2 cancel. Collecting the terms of order ϵ^2 then yields

$$-2\mathbb{P}_2 S^4 + 2\sqrt{f-1}(\mathbb{M}_2 - \mathbb{V}_2 - \sqrt{f})S^2 - (f-1) = 0,$$

where the quantities $\mathbb{P}_2$, $\mathbb{M}_2$ and $\mathbb{V}_2$ are defined in Appendices A and B (see (A 9), (B 6), (B 7) and (B 12)). To leading order in the limit $\varepsilon\sqrt{f} \to 0$, $\gamma - 1 = O(\varepsilon)$, the dispersion relation then takes the form

$$S^4 + 2(\sqrt{f}/\sqrt{f-1})S^2 + 1 = 0, \tag{5.2}$$

which is obtained after the cancellation of the factor $(f-1)$. There are two negative roots of this equation, namely

$$S^2 = -\frac{(\sqrt{f}\pm 1)}{\sqrt{f-1}}, \tag{5.3}$$

and therefore the quantities S and σ are purely imaginary numbers, $\sigma = \mathrm{i}\omega$, $S = \mathrm{i}\omega/\overline{a}_b|k|$, corresponding to two acoustic modes with frequency of the order of $\overline{a}_b|k|$,

$$\frac{\omega}{\overline{a}_b|k|} = \frac{\sqrt{\sqrt{f}\pm 1}}{(f-1)^{1/4}}, \tag{5.4}$$

$\omega = O(\overline{a}_b|k|)$, where the frequency has been taken positive, $\omega > 0$. The + and − signs in (5.3) and (5.4) correspond to two admissible solutions of (5.1). To leading order, the right-hand side of (5.1) is equal to $\omega^2/(\overline{a}_b k)^2$ and $\mathbb{P} = -1$ on the left-hand side. The quantity $\pm\sqrt{\cdots}$ on the left-hand side of (5.1) must, therefore, be equal to $\mp\mathrm{i}\omega/\overline{a}_b|k|$, and, according to (3.1),

$$l = \frac{\omega/\overline{a}_b}{\epsilon\sqrt{f-1}}, \qquad \frac{|k|}{l} = \epsilon\frac{(f-1)^{3/4}}{\sqrt{\sqrt{f}\pm 1}}, \tag{5.5}$$

meaning $|k|/l = O(\epsilon)$.

The following outcomes are worth stressing:

(*a*) The two acoustic modes (plus two other ones, symmetric relatively to the x axis) propagate in a quasi-perpendicular direction to the unperturbed detonation front ($|k|/l \ll 1$). These modes are both upstream running with a relative velocity normal to the unperturbed detonation of order ϵ times the sound speed. This is seen by expanding the quantity in (3.7) and (3.8) for $l > 0$ and $|k|/l$ small

$$u_r \equiv \overline{u}_b - \overline{a}_b \frac{l}{\sqrt{l^2 + k^2}} \approx \overline{a}_b \left[(M_b - 1) + \frac{1}{2} \left(\frac{k}{l} \right)^2 + \cdots \right], \tag{5.6}$$

and by using (A 8) and (5.5), yielding $u_r = -\epsilon \sqrt{f-1}[1 + O(\epsilon^2)]$, which is negative, implying that the detonation wave is stable against disturbances with large wavelength, in the sense of § 3.4. This is consistent with the stability condition against planar disturbances which was required in order to consider that the inner structure of the shock is the same as in the planar case.

(*b*) The perturbed solution in (3.3) and (3.4) is that of a two-length-scale problem: an acoustic wave and a vorticity wave involving, respectively, a short length scale ($2\pi/l$), and a longer one ($2\pi/k$), which is also the typical size of the wrinkles of the shock front.

(*c*) The amplitude of the longitudinal component of the flow velocity in the acoustic wave is of the order of $\partial A/\partial t$ (the perturbation of the front velocity). The amplitude of the transverse velocity is ϵ times smaller. The amplitude of the two components of velocity associated with the vorticity wave is even smaller, ϵ^2 times smaller than $\partial A/\partial t$. This follows directly from (3.2) and (5.5) and from the boundary conditions (B 4), (B 6), (B 8) and (B 11). Because of the boundary conditions of δp_b and δu_b the quantity $\delta u + \delta p/\overline{\rho}_b \overline{u}_b = O(\epsilon \partial A/\partial t)$ vanishes at leading order. The fact that the vorticity wave vanishes up to the first order is also due to (5.5).

(*d*) Clearly, the solution required carrying the perturbation analysis to the second order in ϵ. Qualitatively the same results apply to non-reacting shocks for which a similar analysis may be carried out.

6. Formulation of the problem for unstable disturbances

6.1. *Length and timescales of the unstable disturbances*

We look now for unstable modes of the detonation regimes described in § 4, namely those for which $\mathrm{Re}(\sigma) > 0$ and $l = l_-$ (see § 3.3). The radiation condition at infinity (3.7) then shows that the scaling of the wavelength of these unstable disturbances must be different from that in (5.5). Anticipating $l_-^{(r)} > 0$, the radiation condition (3.7) shows that, according to (5.6) and (A 8), the ratio $|k|/l_-^{(r)}$ cannot be smaller than $\sqrt{\varepsilon}$. The ratio, therefore, must be larger than that in (5.5), and the acoustic waves associated with unstable disturbances propagate in the burnt gas in a more tilted direction than that found in the preceding section. The order of magnitude of the transverse component of the flow velocity (3.4) is therefore also different. A consequence is that the range of the stable modes described in the preceding section (neutral upstream-running modes, $\mathrm{Re}(\sigma) = 0$, with no modification of the detonation structure) cannot be adjacent to the range in the wavenumber coordinate of unstable modes characterized by $\mathrm{Re}(\sigma) > 0$. The transition should occur through a finite-wavelength band of neutral modes involving modifications of the detonation structure and ranging from stable to unstable situations in the sense of the criterion in § 3.4.

Setting

$$|k|/l_-^{(r)} = O(\varepsilon^{1/2}), \tag{6.1}$$

and introducing (A 8) and (6.1) into (3.2), $\omega = \overline{a}_b l_-^{(r)}[(1-M_b) + (1/2)(k/l_-^{(r)})^2 + \cdots]$, we found the acoustic frequency in the burnt gas to be

$$\omega/\overline{a}_b l_-^{(r)} = O(\varepsilon), \qquad \omega/\overline{a}_b k = O(\varepsilon^{1/2}). \tag{6.2}$$

Equations (6.1) and (6.2) correspond to acoustic waves propagating in a direction slightly tilted from the normal direction to the unperturbed front, but more tilted than that in § 5, as mentioned before. With attention focused on instabilities whose linear growth rates are not larger than the frequency of the acoustic waves, the characteristic evolution time is then given by (6.2).

Since Chapman–Jouguet detonations propagate at nearly sonic velocity when the heat release is small, the detonation thickness is determined by the sound speed and the reaction time $d = \overline{a}_u t_r$. The appropriate length scale and time scale are then obtained by anticipating that the shortest length scale in the flow, namely the acoustic wavelength, is not shorter than the detonation thickness. The appropriate scaled coordinates of order unity and the non-dimensional time of order unity, characterizing both the frequency and the growth or the decay rate, are then obtained by the use of (6.1) and (6.2),

$$\xi = (x - A)/(\overline{a}_u t_r), \qquad \eta = \varepsilon^{1/2} y/(\overline{a}_u t_r), \qquad \tau = \varepsilon t/t_r, \tag{6.3}$$

together with

$$\frac{1}{\overline{a}_u}\frac{\partial A}{\partial t} = \varepsilon\frac{\partial \alpha}{\partial \tau}, \qquad \frac{\partial A}{\partial y} = \varepsilon^{1/2}\frac{\partial \alpha}{\partial \eta}, \tag{6.4}$$

where $\alpha = A/(\overline{a}_u t_r)$, with $\partial\alpha/\partial\tau$ and $\partial\alpha/\partial\eta$ being of order unity. Similar to the planar case studied earlier by Clavin & Williams (2002), the slow time scale is associated with the upstream-running acoustic wave, which propagates in the gas in a direction quasi-perpendicular to the detonation front, at a velocity slightly higher than the velocity of the unperturbed gas flowing downstream. This wave controls the longest time in the feedback loop mentioned in § 1. Mass, momentum and energy conservation within the interior of detonation then show that the quantities $u/\overline{a}_u - 1$ and $p/\overline{p}_u - 1$ are both of order ε, and that the ratios $v/\overline{a}_u$ and $T/T_u - 1$ are of order $\varepsilon^{3/2}$ and ε^2, respectively, leading to

$$\mu = 1 + \varepsilon\mu_1, \quad \pi = \varepsilon\pi_1, \quad \nu = \varepsilon^{3/2}\nu_2, \quad \partial v/\partial y = \varepsilon^2 \nu_2', \quad \theta = \varepsilon^2\theta_2, \quad a/\overline{a}_u = 1 + O(\varepsilon^2), \tag{6.5}$$

where the notation of (4.8) has been used, and where μ_1, π_1, ν_2, ν_2' and θ_2 are of order unity and are functions of the reduced variables ξ, η, τ. Expansion of (3.1) gives, at leading order, for $\ell_\pm = l_\pm \overline{a}_u t_r$,

$$\mathrm{i}\ell_\pm = \frac{s \pm \sqrt{2\kappa^2\sqrt{f-1} + s^2}}{2\sqrt{f-1}}, \tag{6.6}$$

where s and κ are non-dimensional quantities of order unity,

$$\sigma = \varepsilon s/t_r, \qquad k = \varepsilon^{1/2}\kappa/(\overline{a}_u t_r). \tag{6.7}$$

6.2. *The leading-order equations*

The ordering of the temperature implies that the sound speed is constant up to terms of order ε^2, $a_u = \overline{a}_u$. Through the first two orders in ε, the variation of the sound speed may, therefore, be neglected in (2.7), giving

$$\left[\frac{\partial}{\partial t}+(\overline{u}+\overline{a}_u)\frac{\partial}{\partial x}\right]\left(\frac{\delta p}{\overline{p}_u}+\frac{\delta u}{\overline{a}_u}\right)+\delta u\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\overline{p}}{\overline{p}_u}+\frac{\overline{u}}{\overline{a}_u}\right)=\varepsilon^2\frac{\delta w}{t_r}-\frac{\partial v}{\partial y}, \tag{6.8}$$

$$\left[\frac{\partial}{\partial t}+(\overline{u}-\overline{a}_u)\frac{\partial}{\partial x}\right]\left(\frac{\delta p}{\overline{p}_u}-\frac{\delta u}{\overline{a}_u}\right)+\delta u\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\overline{p}}{\overline{p}_u}-\frac{\overline{u}}{\overline{a}_u}\right)=\varepsilon^2\frac{\delta w}{t_r}-\frac{\partial v}{\partial y}, \tag{6.9}$$

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\left(\frac{v}{\overline{a}_u}\right)=-\overline{a}_u\frac{\partial}{\partial y}\left(\frac{\delta p}{\overline{p}_u}\right). \tag{6.10}$$

It is worth noticing that the differential operator in the first term on the left-hand side of (6.9) is of order ε,

$$t_r\left[\frac{\partial}{\partial t}+(\overline{u}-\overline{a}_u)\frac{\partial}{\partial x}\right]=\varepsilon\left[\frac{\partial}{\partial \tau}+\left(\overline{\mu}_1-\frac{\partial \alpha}{\partial \tau}\right)\frac{\partial}{\partial \xi}\right], \tag{6.11}$$

and it represents an upstream-running acoustic wave. Equations (6.8), (6.9) and (6.11) are the main simplifications of the limiting case considered here.

The stability analysis is performed in a Fourier decomposition and in the reduced system of coordinates (ξ, η, τ) defined in (6.3) by setting $\alpha=\tilde{\alpha}\mathrm{e}^{\mathrm{i}\kappa\eta+s\tau}$, with κ and s being of order unity. Corresponding field functions ϕ are then written as

$$\phi=\overline{\phi}+\delta\phi, \qquad \delta\phi=\tilde{\phi}(\xi)\tilde{\alpha}\mathrm{e}^{\mathrm{i}\kappa\eta+s\tau}.$$

Use of (A 11) in (6.8) with the boundary conditions (4.6) and (4.7) then yields

$$\frac{\partial}{\partial \xi}(\tilde{\mu}_1+\tilde{\pi}_1)=0, \qquad \tilde{\mu}_1=-\tilde{\pi}_1. \tag{6.12}$$

To leading order in ε, the expansion of (6.9) and (6.10) then gives two coupled equations for the Fourier transforms $\tilde{\mu}_1$ and $\tilde{v}_2'$ of the longitudinal velocity and of the transverse gradient of the transverse velocity, respectively,

$$s\tilde{\mu}_1+\frac{\mathrm{d}}{\mathrm{d}\xi}(\overline{\mu}_1\tilde{\mu}_1)-s\frac{\mathrm{d}}{\mathrm{d}\xi}\overline{\mu}_1=-\frac{\tilde{w}_2}{2}+\frac{\tilde{v}_2'}{2}, \tag{6.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{v}_2'=-\kappa^2\tilde{\mu}_1+\kappa^2\frac{\mathrm{d}}{\mathrm{d}\xi}\overline{\mu}_1, \tag{6.14}$$

where relation (A 12) $d(\overline{\pi}_1+\overline{\mu}_1)=0$ has been used, and where the reaction rate $w_2(\xi,\ \eta,\ \tau)$ is obtained by solving the two equations for Y and θ_2 obtained from the expansion of (2.6), $w_2(\xi,\ \eta,\ \tau)=w(\theta_2,\ Y)$,

$$\frac{\partial}{\partial \xi}Y=w(\theta_2,\ Y), \qquad \frac{\partial}{\partial \xi}\left[\theta_2-\frac{(\gamma-1)}{\varepsilon}\pi_1\right]=w(\theta_2,\ Y), \tag{6.15}$$

where $\gamma=1$ to leading order in ε. Boundary conditions at the shock for (6.13)–(6.15), obtained from (4.6) and (4.7), are

$$\xi=0: \qquad \tilde{\mu}_1=2s, \quad \tilde{v}_2=-2\kappa^2\sqrt{f}, \quad Y=0, \quad \tilde{\theta}_2=-2\frac{(\gamma-1)}{\varepsilon}s. \tag{6.16}$$

Eliminating $\tilde{\mu}_1$ from (6.13) and (6.14) leads to a second-order ordinary differential equation for $\tilde{v}_2'$ with two boundary conditions at the shock,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(q\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{v}_2'\right) - s\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{v}_2' - \frac{\kappa^2}{2}\tilde{v}_2' = -\kappa^2\frac{\mathrm{d}}{\mathrm{d}\xi}\left(q\frac{\mathrm{d}}{\mathrm{d}\xi}q\right) - \frac{\kappa^2}{2}\tilde{w}_2, \tag{6.17}$$

$$\xi = 0: \qquad \tilde{v}_2 = -2\kappa^2\sqrt{f}, \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{v}_2' = -\kappa^2\left(2s + \frac{\mathrm{d}}{\mathrm{d}\xi}q\right), \tag{6.18}$$

where $q(\xi) = -\overline{\mu}_1(\xi)$ is a positive function decreasing from $\sqrt{f}$ at $\xi = 0$ to $\sqrt{f-1}$ at $\xi \to +\infty$. This function represents the inner structure of the unperturbed planar detonation,

$$2q\frac{\mathrm{d}}{\mathrm{d}\xi}q = -\overline{w}_2, \quad \int_0^{+\infty}\overline{w}_2(\xi)\,\mathrm{d}\xi = 1, \quad q(0) = \sqrt{f}. \tag{6.19}$$

6.3. *The downstream boundary condition*

The extra condition that has to be used for determining the unknown $s(\kappa)$ is the boundedness condition in the burnt gas where the reaction has gone to completion,

$$\xi \to +\infty: \qquad \tilde{v}_2 \propto \mathrm{e}^{\mathrm{i}\ell - \xi}. \tag{6.20}$$

Equation (3.6) (incompressibility of the vorticity wave) is not helpful here. To leading order, this condition yields

$$\xi \to +\infty: \qquad \left(s + \sqrt{2\kappa^2\sqrt{f-1} + s^2}\right)\tilde{\mu}_1 = \tilde{v}_2', \tag{6.21}$$

where, according to (6.14), $\tilde{\mu}_1 = -(\mathrm{d}\tilde{v}_2'/\mathrm{d}\xi)/\kappa^2$, the unperturbed flow being uniform in the burnt gas, $\mathrm{d}\overline{\mu}_1/\mathrm{d}\xi = 0$. Equation (6.21) is automatically satisfied by the solution of (6.17) in the burnt gas, where $\tilde{v}_2 \propto \mathrm{e}^{\mathrm{i}\ell - \xi}$. This is an illustration of the fact that the perturbation of the flow in the burnt gas is a purely acoustic wave to leading order in ε, $\delta\pi_1 + \delta\mu_1 = 0$. In other words, the flow involved in the vorticity wave is of next order in the asymptotic analysis, since non-zero constant terms are excluded from the solutions $\tilde{\mu}_1(\xi)$ and $\tilde{v}_2'(\xi)$ of (6.12)–(6.18). To leading order in ε, the flow velocity varies on the short length scale only (see also the expansion of (6.8) and (6.9) in the burnt gas).

The problem defined by (6.17)–(6.20) is closed when the heat-release rate $\tilde{w}_2(\xi)$, the solution of (6.15), is known. However, (6.15) is coupled to (6.17) through compressibility effects which introduce the pressure term in energy conservation, $\tilde{\pi}_1 = -\tilde{\mu}_1$, the second equation in (6.15). Therefore, the heat-release-rate perturbations strictly cannot be determined independently of the flow. A simplification, however, is provided by an approximation which is well verified in real detonations. Careful attention, therefore, needs to be paid to selection of appropriate approximations for the perturbation of the heat-release rate, as will now be demonstrated.

7. The heat-release rate

7.1. *General considerations*

Since $Y = 0$ at $\xi = 0$ and $Y \to 1$ at $\xi \to \infty$, spatial integration of the first in (6.15) readily shows that

$$\int_0^{\infty} w_2\,\mathrm{d}\xi = 1, \qquad \int_0^{\infty}\tilde{w}_2\,\mathrm{d}\xi = 0. \tag{7.1}$$

This is true in general, provided that chemical equilibrium is reached at the exit of the detonation structure, the total chemical heat release being constant.

If the perturbed heat-release rate $\tilde{w}_2$ was a specified function of ξ, then the problem defined by (6.17)–(6.20) would become a second-order linear problem with three boundary conditions and thereby may determine a dispersion condition $s(\kappa)$ that must be satisfied for a solution to exist. This would have been the case if, for example, there were no pressure term in (6.15). However, this term cannot be omitted on the basis of an order-of-magnitude analysis.

Equation (6.15) describes a problem which is of third order when the heat-release-rate perturbations depend on the temperature perturbations, that is, $\delta w_2(\xi)$ involves $\delta\theta(\xi)$. If, in addition, it depends on other variables, such as species concentrations that obey differential equations, then the problem would be of higher than third order. A more general state-dependent heat-release rate would have $w(T, p, Y)$, a specific nonlinear function of local state variables, temperature, pressure, Y and so on. On the basis of the realization that the reaction rates in gaseous detonations typically are more strongly dependent on temperature than on pressure or composition, orderings generally are introduced rendering the last two effects negligible compared with the first. This leads in perturbation to $\delta w_2(\xi) = B\overline{w}'_T(\xi)\delta\theta_2(\xi)$, where $\overline{w}'_T(\xi)$ is proportional to the partial derivative of the reaction rate with respect to the temperature $\partial w/\partial T$ computed in the unperturbed planar detonation, and B is a parameter describing the sensitivity of the reaction rate to the temperature. This is typical of what is obtained through activation-energy asymptotics and necessitates treating energy conservation (6.15) along with the rest of the problem.

The previous analysis for the limit of strong overdrive by Clavin & He (1996*b*), Clavin *et al.* (1997) and Daou & Clavin (2003) essentially selected this description, but, in that limit, the pressure term in (6.15) was not present in this equation, because the compressibility effect that it accounts for in energy conservation is of higher order in the low-speed flow encountered at strong overdrive in the Newtonian limit. However, an unsteady term, which arises as a consequence of the entropy-wave convection behind the shock, was included in (6.15). A consequence was that the heat-release-rate perturbation depended not only on ξ but also on the temperature perturbation at the Neumann state, evaluated at an earlier time $\tau - \xi$. This delay effect resulted in the problem finally involving integral equations.

In the present problem, the convective delay is negligibly short, and so the heat release responds quasi-steadily to temperature variations. Here the delay is associated with slow upstream acoustic-wave propagation. However, as already mentioned, the compressibility effects on the reaction rate cannot be neglected. A model for the chemistry in realistic detonations helps to overcome this difficulty, as now explained.

7.2. *Approximations for the heat-release rate*

A helpful simplification is to assume that the heat-release-rate perturbations depend only on the Neumann-state temperature fluctuations and are insensitive to temperature variations elsewhere. This results in $\tilde{w}_2 = b\overline{w}'_N(\xi)\tilde{\theta}_2(\xi = 0)$, where b is a positive parameter describing the sensitivity of the reaction rate to the Neumann temperature and $\overline{w}'_N(\xi)$ is a function of order unity which satisfies $\int_0^\infty \overline{w}'_N(\xi)\,\mathrm{d}\xi = 0$. The non-dimensional perturbation of the Neumann temperature $\theta_2(\xi = 0)$ is given by (6.16), so that

$$\tilde{w}_2 = -h\overline{w}'_N(\xi)s, \qquad \int_0^\infty \overline{w}'_N(\xi)\,\mathrm{d}\xi = 0, \qquad h = 2b(\gamma - 1)/\varepsilon, \tag{7.2}$$

where h is a positive scalar.

With this simplification, the problem is reduced to the second-order problem defined in (6.17)–(6.20), because (6.15) for energy conservation, taking into account the compressibility effects, may be then ignored initially and used only to obtain the perturbation of the temperature profile afterwards.

An approximation of the type of (7.2) has been introduced previously from a purely formal point of view, namely simply defining reaction-rate perturbations to be proportional to the Neumann temperature rather than to the local temperature. Equation (7.2) can, however, be afforded better justification from the viewpoint of the chemistry that occurs in real detonations.

Real gaseous detonations have a strongly temperature-dependent induction zone involving chain initiation and branching. This induction zone is at most weakly exothermic and is followed by strongly exothermic but quasi-temperature-insensitive recombination reactions effecting chain termination in the principal heat-release zone. In addition, in conditions of real detonations, the flow in the induction zone is sufficiently subsonic that it is not sensitive to the compressibility-induced fluctuations of temperature. With this chemistry, the induction zone everywhere is at the Neumann temperature, and so it is only that temperature, and not the temperature later, that affects the chemical heat-release rate. Good physical justifications thus can be given for the selection made in (7.2).

As an explicit model of this type, the induction zone my be considered to be energetically neutral, to persist for a temperature-dependent induction time and to be followed by a heat-release zone in which the reaction rate is independent of temperature. In the well-known square-wave model, the reaction takes place instantaneously after the induction period and results in $w'_N(\xi)=\delta'(\xi-\xi_r)$, where δ' denotes the derivative of the Dirac delta function. If instead the reaction rate is constant during a finite time until the reactants are depleted, this model results in $\overline{w}'_N(\xi)=\delta(\xi-\xi_i)-\delta(\xi-\xi_r)$, where δ denotes the Dirac delta function, ξ_i and ξ_r $(\xi_i<\xi_r)$ being respectively the non-dimensional induction length and detonation thickness of the unperturbed planar solution. If, after induction, the reaction rate instead decays monotonically with a characteristic fixed time, then

$$\overline{w}'_N(\xi)=\delta(\xi-\xi_i)-f(\xi-\xi_i)H(\xi-\xi_i), \tag{7.3}$$

where $H(\xi)$ and $f(\xi)$ denote, respectively, the Heaviside step function and a normalized positive function, $f>0$, $\int_0^\infty f(\xi)\,\mathrm{d}\xi=1$.

Generally speaking, the dynamics of the detonation is obtained by solving integral equations (see Clavin *et al.* 1997 for the strongly overdriven regimes and § 8.1 below for the neighbourhood of the Chapman–Jouguet condition). Discontinuous models, as those mentioned above, usually lead to pathological dynamics in integral-equation formulations (see § 8.4 for the regimes considered here). Continuous models for the heat-release rate, therefore, have to be considered to develop well-behaved solutions. In real detonations, the temperature increases continuously, slightly at the beginning of the induction zone, strongly at the end of the induction zone and finally smoothly up to the end of the reaction zone. Typical functions $\overline{w}'_N(\xi)$ in real detonations vanish at the Neumann state and also at a point $\xi=\xi_i$, as well as in the burnt gas, $\overline{w}'_N(0)=\overline{w}'_N(\xi_i)=\overline{w}'_N(\infty)=0$. They are positive for $0<\xi<\xi_i$ and negative for $\xi>\xi_i$. The continuous model that was introduced earlier in the study of strongly overdriven regimes (see Clavin & He 1996*b*; Clavin *et al.* 1997),

$$\overline{w}'_N(\xi)=\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi\overline{w}), \qquad \overline{w}(\xi)=\frac{m^{n+1}}{n!}\xi^n\mathrm{e}^{-m\xi}, \tag{7.4}$$

is useful as a specific example for studying the transition from stable to unstable situations (see § 8.3). The details of the linear dynamics of the detonation depend on the specific shape of the function $\overline{w}'_N(\xi)$, but the general trends should be well represented, at least qualitatively, by model (7.4).

8. Solution and discussion of the results

8.1. *The general dispersion relation*

The problem addressed initially is obtained from (6.17)–(6.20) with (7.2). It is convenient to employ

$$\zeta = \int_0^{\xi} \frac{d\xi}{q} \tag{8.1}$$

as the independent variable and to introduce the notation

$$Z(\zeta) = \tilde{\nu}'_2(\xi), \quad R(\zeta) = q(\xi), \quad F(\zeta) = hq(\xi)\overline{w}'_N(\xi), \quad \int_0^{+\infty} F(\zeta)\,d\zeta = 0, \tag{8.2}$$

where $Z(\zeta)$ is an unknown function, while $R(\zeta)$ and $F(\zeta)$ are given functions of ζ characterizing the unperturbed planar detonation, the zero integral in (8.2) coming from (7.2). The functions $R(\zeta)$ and $F(\zeta)$ both depend on properties of the heat-release-rate function. According to (6.19), $R(\zeta)$ is a positive monotonically decreasing function with $R(0) = \sqrt{f}$, $\lim_{\zeta\to+\infty} R(\zeta) = \sqrt{f-1}$, representing the density profile of the unperturbed planar detonation structure, while $F(\zeta)$ describes the sensitivity of the distribution of heat-release-rate to the Neumann temperature. It characterizes, therefore, the deformation of the planar detonation structure when varying the overdrive factor for a fixed heat release. In this manner, it is found that

$$\frac{d^2}{d\zeta^2}Z - s\frac{d}{d\zeta}Z - \frac{\kappa^2}{2}RZ = \kappa^2 S, \qquad S = -\left(\frac{d^2}{d\zeta^2}R - \frac{s}{2}F\right), \tag{8.3}$$

$$\zeta = 0: \qquad Z = -2\kappa^2\sqrt{f}, \quad \frac{d}{d\zeta}Z = -\kappa^2\left(2s\sqrt{f} + \frac{d}{d\zeta}R\right), \tag{8.4}$$

$$\zeta \to +\infty: \ \ Z \propto e^{i\zeta L_{b-}}, \tag{8.5}$$

with by definition

$$i l_{b\pm} = i\ell_{\pm}\sqrt{f-1} = \frac{s \pm \sqrt{2\kappa^2\sqrt{f-1} + s^2}}{2}. \tag{8.6}$$

Equations (8.3)–(8.5) determine s as a function of κ. Equations (8.4) at the Neumann state are the Rankine–Hugoniot conditions, and (8.5) ensures that the acoustic waves associated with unstable disturbances, $\mathrm{Re}(s) > 0$, are bounded at infinity.

The physical interpretation is clear. The left-hand side of the first (8.3) represents upstream-running acoustic waves propagating across the detonation structure in a direction quasi-perpendicular to the unperturbed front. The source term $\kappa^2 S$ on the right-hand side results from perturbation of the heat-release rate through front wrinkling. The first term in S comes from the quasi-steady modification of the detonation structure, while the second term represents unsteady effects. The boundedness condition at infinity (8.5) corresponds to a radiative condition in the burnt gas, as discussed previously.

The solution may be written in terms of two particular solutions Φ_+ and Φ_- of the homogeneous equation, $d^2\Phi/d\zeta^2 - s\,d\Phi/d\zeta - \kappa^2 R\Phi/2 = 0$, satisfying the two boundary conditions

$$\lim_{\zeta\to+\infty} \Phi_\pm \propto \mathrm{e}^{\mathrm{i}\zeta L_{b\pm}} \quad \text{and} \quad \Phi_\pm(\zeta=0)=1. \tag{8.7}$$

The solution to (8.3) satisfying the boundedness condition (8.5) at infinity in the burnt gas may then be written in terms of the boundary value at the shock, $Z(0)$,

$$Z(\zeta)=\left[Z(0)+\kappa^2\int_0^\infty \mathrm{d}\zeta\,\frac{\Phi_- S}{W}-\kappa^2\int_0^\zeta \mathrm{d}\zeta\,\frac{\Phi_+ S}{W}\right]\Phi_-(\zeta)-\Phi_+(\zeta)\kappa^2\int_\zeta^\infty \mathrm{d}\zeta\,\frac{\Phi_- S}{W}, \tag{8.8}$$

where $W=\Phi_-\Phi_+'-\Phi_+\Phi_-'$ is the Wronskian, Φ' denoting the derivative with respect to ζ, $\Phi'=\mathrm{d}\Phi/\mathrm{d}\zeta$. The derivative at the Neumann state $Z'(0)\equiv \mathrm{d}Z/\mathrm{d}\zeta|_{\zeta=0}$ as obtained from (8.8), is

$$Z(0)_{\zeta=0}=Z(0)\Phi_-'(0)+\left[\Phi_-'(0)-\Phi_+'(0)\right]\kappa^2\int_0^\infty \mathrm{d}\zeta\,\frac{\Phi_- S}{W}. \tag{8.9}$$

The dispersion relation is then obtained when the expressions $Z(0)$ and $Z'(0)$ in (8.4) together with the definition of S in (8.3) are introduced into (8.9). If it is assumed that the heat-release rate is negligible at the Neumann state, then $R'(0)\equiv \mathrm{d}R/\mathrm{d}\zeta|_{\zeta=0}=0$. Since generally this is the case in real detonations,

$$R'(0)=0: \qquad \Phi_\pm'(0)=\mathrm{i}l_{N\pm}, \quad \mathrm{i}l_{N\pm}=\frac{s\pm\sqrt{2\kappa^2\sqrt{f}+s^2}}{2}, \tag{8.10}$$

the dispersion relation then takes the form

$$\sqrt{f}\left(s+\sqrt{2\kappa^2\sqrt{f}+s^2}\right)=\sqrt{2\kappa^2\sqrt{f}+s^2}\int_0^\infty \mathrm{d}\zeta\,\frac{\Phi_- S}{W}. \tag{8.11}$$

This dispersion relation is valid for $\mathrm{Re}(s)>0$ and for any value of $f\geqslant 1$ satisfying $\varepsilon^2 f\ll 1$. The Chapman–Jouguet detonation corresponds to $f=1$.

Note that there is no solution to the dispersion relation when the source term S is neglected. This is consistent with the fact that, according to § 6.1, the scaling laws for the unstable linear modes, $\mathrm{Re}(s)>0$, are different from those at large wavelength when the modifications of the distribution of the heat-release rate are negligible. In other words, the instabilities which are characterized by $\mathrm{Re}(s)>0$ always result from the modifications of the inner structure of the detonation.

Equation (8.11) is a dispersion relation determining s when κ is given. This general dispersion relation, however, is very complicated. Because of the spatial variation of $R(\zeta)$, explicit analytical expressions for Φ_-, Φ_+ and W cannot be written in the general case. To explore properties of the dispersion relation further it is helpful to introduce additional approximations.

8.2. *Simplification at moderate overdrive*

Study of the case of moderate overdrive can enhance understanding of the multidimensional problem. By assuming that f is a sufficiently large number, the variation of Q across the detonation structure, from $\sqrt{f}$ to $\sqrt{f-1}$, may be neglected in the equations for Φ_- and Φ_+. The solutions satisfying (8.7) then become

$$\Phi_\pm\approx \mathrm{e}^{\mathrm{i}\zeta L_{N\pm}}, \qquad W\approx \mathrm{e}^{\zeta s}\sqrt{2\kappa^2\sqrt{f}+s^2}. \tag{8.12}$$

This limit still corresponds to slightly overdriven regimes near Chapman–Jouguet conditions for small heat release, provided that the quantity $\varepsilon^2 f$ is a small number. This limit leads to the simplest model for studying the coupling of the acoustic waves and heat-release variations in the inner structure, which is involved in the dynamics of gaseous detonations. The acoustic waves propagate in a quasi-uniform medium, and the feedback loop mentioned in the introduction may, therefore, be easily described in simple mathematical terms with this approximation.

When the second spatial derivative of R is not singular, the first term in (8.3) for S becomes negligible, because it is proportional to $\sqrt{f}-\sqrt{f-1}$ and (8.11), written with the ξ-variable, then yields

$$s+\sqrt{2\kappa^2\sqrt{f}+s^2}=s\frac{h}{2\sqrt{f}}\int_0^{\infty}\overline{w}'_N(\xi)\mathrm{e}^{-\xi(s+\sqrt{2\kappa^2\sqrt{f}+s^2})/2\sqrt{f}}\mathrm{d}\xi. \tag{8.13}$$

The plus sign in front of the square root in the exponent on the right-hand side of (8.13) comes from the term $\mathrm{e}^{\mathrm{i}\zeta L_{N-}}/\mathrm{e}^{\zeta s}$ on the right-hand side of (8.11). After introduction of the notation

$$s'\equiv s/\sqrt{f},\qquad \kappa'^2\equiv 2\kappa^2/\sqrt{f},\qquad h'\equiv h/(2\sqrt{f}), \tag{8.14}$$

(8.13) reads

$$s'+\sqrt{\kappa'^2+s'^2}=s'h'\int_0^{+\infty}\overline{w}'_N(\xi)\mathrm{e}^{-\xi\left(s'+\sqrt{\kappa'^2+s'^2}\right)/2}\mathrm{d}\xi, \tag{8.15}$$

where s' and κ' are the reduced complex growth rate and the reduced wavenumber, respectively, and where h' is a parameter that increases with increasing temperature sensitivity or with approach to Chapman–Jouguet conditions (decreasing the overdrive factor f).

For a given wavenumber κ', the dispersion relation (8.15) is an equation for the complex growth rate $s'(\kappa')$ involving a single scalar parameter h'.

8.3. *Analytical expressions at bifurcation*

The function $\overline{w}'_N(\xi)$ represents the deformation of the distribution of the heat-release rate. Its general shape was discussed at the end of § 7.2. For a given smooth function $\overline{w}'_N(\xi)$, (8.15) describes a Hopf bifurcation which occurs upon increasing the ratio $h/\sqrt{f}$, as may be checked numerically. Below a critical value there is no solution with $\mathrm{Re}(s')>0$, while a narrow band of unstable modes around a finite wavelength appears just above the critical value. The regime of cellular detonation thus appears either by approaching Chapman–Jouguet conditions (decreasing f) or by increasing the thermal sensitivity of the reaction rate (increasing h). For a given value of the ratio $h/\sqrt{f}$, the bifurcation occurs also by stiffening the function $\overline{w}'_N(\xi)$. The stiffer the function $\overline{w}'_N(\xi)$ is, the smaller is the critical value of the ratio $h/\sqrt{f}$ at which the cellular structures appear, and the larger are the frequency of oscillation and the wavenumber of the cellular structures.

The study of the bifurcation may be carried out analytically for the particular example in (7.4), with n being an integer, for which the Laplace transform may be written explicitly, $\int_0^{\infty}\overline{w}'_N(\xi)\mathrm{e}^{-\xi z}\mathrm{d}\xi=(n+1)(z/m)\,[1+(z/m)]^{-(n+2)}$. In this case, (8.15) becomes a polynomial equation for the reduced complex linear growth rate s',

$$\left[2+\frac{s'+\sqrt{\kappa'^2+s'^2}}{m}\right]^{(n+2)}=\frac{\beta^{n+2}s'}{m},\qquad \beta^{n+2}\equiv\frac{(n+1)2^n h}{\sqrt{f}}. \tag{8.16}$$

The stiffness of the induction zone and of the heat-release zone increases when n or m is increased. The ratio n/m measures the induction length. Equation (8.16) may be investigated without loss of generality by taking $m=1$, because if $m\neq 1$ it is merely necessary to replace s' and κ' by s'/m and κ'/m, respectively. We are, thus, left with a polynomial equation for $s'(\kappa')$ containing two parameters, n and β, characterizing the stiffness of the induction zone and the sensitivity to temperature, respectively.

The method for determining the bifurcation point is classical. Introducing purely imaginary roots $s' = \pm i\omega'$ into (8.16) with $\omega' > 0$, one gets two equations relating to the three real quantities ω', κ' and β, one equation for the real part of (8.16) and the other for the imaginary part. These two equations are compatible only for a particular relation between the bifurcation parameter β and the wavenumber κ'. The bifurcation is then obtained as the minimum of the parameter β for which a solution exists.

Two cases must be considered separately, depending on the relative value of ω' and κ':

$$\text{case I, } \kappa' > \omega' : \qquad (2 + \sqrt{\kappa'^2 - \omega'^2}) = \beta\omega'^{(1/(n+2))}\mathrm{Re}[(\pm i)^{(1/(n+2))}] \tag{8.17}$$

$$\pm\omega' = \beta\omega'^{(1/(n+2))}\mathrm{Im}[(\pm i)^{(1/(n+2))}] \tag{8.18}$$

$$\text{case II, } \kappa' < \omega' : \quad \pm(\omega' + \sqrt{\omega'^2 - \kappa'^2}) = \beta\omega'^{(1/(n+2))}\mathrm{Im}[(\pm i)^{(1/(n+2))}] \tag{8.19}$$

$$2 = \beta\omega'^{(1/(n+2))}\mathrm{Re}[(\pm i)^{(1/(n+2))}]. \tag{8.20}$$

On the left-hand side of (8.19), the plus sign in the parenthesis is selected because we focus our attention on the unstable situations for which $\mathrm{Re}(s') \geqslant 0$ with a boundedness condition at infinity. This may be explained as follows. Consider a weakly unstable case, adjacent to the marginal case, $s' = \pm i\omega' + \epsilon'$, with $0 < \epsilon' \ll \omega'$ and $\omega' > \kappa'$. The $+$ sign in front of the square root in the bracket on the left-hand side of (8.16), meaning that the real part of the square root must be positive, imposes the $+$ sign in (8.19), as may be verified to first order in the expansion in small ϵ',

$$\sqrt{s'^2 + \kappa'^2} \approx \sqrt{-(\omega'^2 - \kappa'^2)}\left[1 - \frac{(\pm i\omega')\epsilon'}{(\omega'^2 - \kappa'^2)}\right] \quad \Rightarrow \quad -(\pm i\omega')\sqrt{-(\omega'^2 - \kappa'^2)} > 0.$$

The $\pm$ sign in front of $i\omega'$ must be therefore the same as that in front of $i\sqrt{\omega'^2 - \kappa'^2}$, as written in (8.19).

According to (8.17) and (8.20), the only roots among the $2(n+2)$ roots, $(+i)^{(1/(n+2))}$ and $(-i)^{(1/(n+2))}$, that are relevant are those having a positive real part. The computation is the same for all of them. Let us consider a particular one $e^{i\theta_l}$ with $0 < \theta_l < \pi/2$, the case $-\pi/2 < \theta_l < 0$ being treated in the same way.

Consider the first case I, $\omega' < \kappa'$. The bifurcation parameter β may be eliminated from (8.17) and (8.18), yielding

$$\frac{\omega'}{t_l} - 2 = \sqrt{\kappa'^2 - \omega'^2}, \qquad t_l \equiv \tan\theta_l > 0, \tag{8.21}$$

which provides us with an expression for ω' in terms of κ' when $\mathrm{Re}(s') = 0$,

$$\kappa' > 2t_l : \quad \frac{\omega'}{t_l} = \frac{2t_l + \sqrt{4 + (\kappa'^2 - 4)\left(1 + t_l^2\right)}}{\left(1 + t_l^2\right)}. \tag{8.22}$$

This solution, which corresponds to $\omega' < \kappa'$, is valid only for $\kappa' > 2t_l$ in order to satisfy the condition $\omega' > 2t_l$ which is imposed by (8.21). There is no solution to (8.21) for $\kappa' < 2t_l$. Note also that the straight line $\omega' = \kappa'$ in the plan (κ', ω') is tangent at the point $(\kappa' = 2t_l, \omega' = 2t_l)$ to the hyperbolic curve $\omega'(\kappa')$ corresponding to (8.22), the curve being below the tangent elsewhere for $\kappa' > 0$. According to (8.18), the bifurcation parameter β may be expressed in term of ω' as

$$\beta = \frac{\omega'^{(n+1)/(n+2)}}{\sin\theta_l}, \qquad \beta_m = \frac{(2t_l)^{(n+1)/(n+2)}}{\sin\theta_l}, \tag{8.23}$$

and the minimum value β_m of β in this branch of solution corresponds to $\kappa' = 2t_l$ and $\omega' = 2t_l$.

Now consider case II. Eliminating β yields

$$\omega' - 2t_l = -\sqrt{\omega'^2 - \kappa'^2}, \tag{8.24}$$

whose solution is

$$\kappa' < 2t_l : \quad \omega' = t_l + \frac{\kappa'^2}{4t_l}, \tag{8.25}$$

which is valid only for $\kappa' < 2t_l$ since, according to (8.24), $\omega' < 2t_l$. The straight line $\omega' = \kappa'$ is also tangent at the point ($\kappa' = 2t_l$, $\omega' = 2t_l$) to the parabolic curve $\omega'(\kappa')$ corresponding to (8.25), the curve being above the tangent elsewhere. According to (8.20),

$$\beta = \frac{2}{\cos\theta_l}\frac{1}{\omega'^{(1/(n+2))}}, \qquad \beta_m = \frac{(2t_l)^{(n+1)/(n+2)}}{\sin\theta_l}, \tag{8.26}$$

and the minimum of β for case II corresponds to the same point as in case I, $\kappa' = 2t_l$ and $\omega' = 2t_l$, leading to the same minimum value β_m.

A linear expansion $s' \to i\omega' + \delta s'$, $\kappa' \to \kappa' + \delta\kappa'$ then shows that $\mathrm{Re}(\delta s') > 0$ corresponds to $\delta\kappa' < 0$ in case I and to $\delta\kappa' > 0$ in case II. This shows that, for a fixed θ_l, unstable solutions with $\mathrm{Re}(s') > 0$ bifurcate when β is increased above β_m. This last quantity varies with θ_l. The critical value β_c of the bifurcation parameter associated with the onset of the instability is, therefore, the minimum of all the values of β_m belonging to the finite set associated with all the possible values of θ_l. Introducing the notation θ_{lc} for this particular value, one gets the following critical values:

$$\beta_c = \frac{(2\tan\theta_{lc})^{(n+1)/(n+2)}}{\sin\theta_{lc}}, \qquad \kappa'_c = 2\tan\theta_{lc}, \qquad \omega'_c = 2\tan\theta_{lc}, \tag{8.27}$$

or, when written in the original variables according to (8.14) and (8.16),

$$\left(\frac{h}{\sqrt{f}}\right)_c = \frac{(2\tan\theta_{lc})^{n+1}}{(n+1)2^n(\sin\theta_{lc})^{n+2}}, \quad \kappa_c = m\sqrt{2}f^{1/4}\tan\theta_{lc}, \quad \mathrm{Im}(s_c) = 2m\sqrt{f}\tan\theta_{lc}, \tag{8.28}$$

yielding, according to (6.7),

$$k_c = \varepsilon^{1/2}\frac{n\sqrt{2}f^{1/4}\tan\theta_{lc}}{\bar{a}_u t_i}, \qquad \omega_c = \varepsilon\frac{2n\sqrt{f}\tan\theta_{lc}}{t_i}, \tag{8.29}$$

where $t_i = (n/m)t_r$ is the induction time, k_c and ω_c being the critical wavenumber and the critical frequency, $\omega = \mathrm{Im}(\sigma)$.

These results describe how the critical thermal sensitivity h_c decreases as the Chapman–Jouguet regime is approached (decreasing f towards unity), the key dependence being that h_c is proportional to $\sqrt{f}$. Correspondingly, increasing the overdrive factor promotes the stability of the detonation. Another outcome in (8.29) is the dependence of the critical wavenumber and the critical frequency on f, showing that they decrease as the Chapman–Jouguet regime is approached.

8.4. *Pathological dynamics for singular distribution of the heat-release rate*

The function $\overline{w}'_N(\xi)$ in (7.4) becomes stiffer and stiffer when n and m are increased. The function becomes singular with a finite induction length in the double limit $n \to \infty$, $m \to \infty$, n/m fixed. The angle θ_{lc} which is involved in the critical value β_c in (8.27), decreases as $1/\sqrt{n}$, when n is increased. According to (8.28), the critical value $(h/\sqrt{f})_c$, therefore, decreases like $1/\sqrt{n}$, while the critical frequency and wavenumber increase like $\sqrt{n}$. This shows how increasing the stiffness of the distribution of the heat-release rate promotes the instability.

In the singular limit $n \to \infty$, $m \to \infty$, n/m fixed, the dynamics of the detonation becomes pathological since instabilities, involving infinite growth rate, infinite frequencies and infinite wavenumbers, systematically develop, whatever be the thermal sensitivity h and/or the overdrive factor f.

A similar pathology, but with a non-zero critical value of the bifurcation parameter, is observed with the discontinuous model (7.3) with $f(\xi) = \mathrm{e}^{-\xi}$. Equation (8.15) yields

$$\left[1 + \frac{(s' + \sqrt{s'^2 + \kappa'^2})}{2}\right] = \frac{s'h}{4\sqrt{f}}\mathrm{e}^{-\left(s' + \sqrt{s'^2+\kappa'^2}\right)\xi_i/2}. \tag{8.30}$$

Looking for the bifurcation by the same method as before, and focusing first our attention on the case $\omega' > \kappa'$, we are led to solve the system of equations

$$1 = \frac{h}{4\sqrt{f}}\omega' \sin\left[\frac{(\omega' + \sqrt{\omega'^2 - \kappa'^2})}{2}\xi_i\right], \tag{8.31}$$

$$\frac{(\omega' + \sqrt{\omega'^2 - \kappa'^2})}{2} = \frac{h}{4\sqrt{f}}\omega' \cos\left[\frac{(\omega' + \sqrt{\omega'^2 - \kappa'^2})}{2}\xi_i\right]. \tag{8.32}$$

Eliminating the bifurcation parameter leads to the equation $\xi_i/z = \tan z$ for $z = (\omega' + \sqrt{\omega'^2 - \kappa'^2})\xi_i/2$ with, according to (8.31), $\sin z > 0$. This leads to an unbounded countable set of increasing values of z, $z_n = 2\pi n + r_n$, with $\pi/2 > r_n \geqslant 0$, r_n decreasing to zero as the integer n increases. For a given n, the function ω' of κ' is the part of the parabola $\omega'(\kappa') = (z_n/\xi_i) + \kappa'^2/(4(z_n/\xi_i))$ limited by the two points $(\kappa' = 0, \omega' = z_n/\xi_i)$ and $\kappa' = \omega' = 2z_n/\xi_i$, in order to satisfy the conditions $\kappa \geqslant 0$ and $\omega' \leqslant 2z/\xi_i$. Therefore, according to (8.32), the bifurcation parameter $h/\sqrt{f}$ reaches its minimum value 2 when $\omega' = \kappa'$ for $\cos z \to 1$, which corresponds to infinite frequencies and wavenumbers. The same minimum value of the bifurcation parameter, with the same pathology, is observed in the case $\omega' < \kappa'$,

$$1 + \frac{\sqrt{\kappa'^2 - \omega'^2}}{2} = \frac{h}{4\sqrt{f}}\omega' \mathrm{e}^{-\xi_i(\sqrt{\kappa'^2 - \omega'^2})/2} \sin(\omega'\xi_i/2), \tag{8.33}$$

$$\frac{1}{2} = \frac{h}{4\sqrt{f}}\mathrm{e}^{-\xi_i(\sqrt{\kappa'^2 - \omega'^2})/2} \cos(\omega'\xi_i/2), \tag{8.34}$$

yielding

$$1 + \frac{\sqrt{\kappa'^2 - \omega'^2}}{2} = \frac{h}{4\sqrt{f}}\omega' \tan(\omega'\xi_i/2). \tag{8.35}$$

According to (8.34), the bifurcation parameter must satisfy $h/\sqrt{f} \geqslant 2$, and the minimum value 2 corresponds to infinite frequencies and wavenumbers.

9. Conclusions and perspectives

The multidimensional stability of gaseous detonations near Chapman–Jouguet conditions has been investigated for small heat release. This study is complementary to our 1997 analysis of strongly overdriven detonations for which, according to a Newtonian approximation, the cellular structures result from an instability in the coupling of the vorticity–entropy wave with the heat-release rate (see Clavin *et al.* 1997). For the conditions investigated in the present paper, the instability results from the coupling of the acoustic waves with the heat-release rate, the entropy–vorticty wave playing a negligible role at leading order. The end result assumes, however, a similar form to that of our previous strong-overdrive result, namely an integral equation can be obtained for the linear growth rate involving a function of space $\overline{w}'_N(\xi)$ representing the modification of the distribution of heat-release rate of the unperturbed planar detonation when varying the overdrive. Equation (8.11) is a dispersion relation for unstable detonations with a positive real part of the linear growth rate, $\mathrm{Re}(\sigma) > 0$, valid also at the Champan–Jouguet conditions for any chemistry. Note that, as explained in § 6.1, detonations with $\mathrm{Re}(\sigma) = 0$, unstable in the sense discussed in § 3.4, may also exist to fill the finite-wavelength gap joining the stable disturbances with $\mathrm{Re}(\sigma) = 0$ at long wavelength exhibited in § 5.

The integral equation becomes simpler at moderate overdrive, still close to the Chapman–Jouget conditions, as seen in (8.15). A polynomial (8.16) for the dispersion relation was then obtained for a model of distribution $\overline{w}'_N(\xi)$ presented in (7.4). The result shows the existence of a Hopf bifurcation point at a finite wavelength, the characteristics of which are given in (8.28) and (8.29) in which the stiffness of the induction zone and of the exothermic zone increases with n and m, the thickness of the induction zone being proportional to the ratio n/m. For a given shape $\overline{w}'_N(\xi)$, that is for n and m fixed, there is a single scalar bifurcation parameter $h/\sqrt{f}$ grouping the scaled overdrive factor f defined in (4.2) and the sensitivity of the heat-release rate to temperature h of (7.2). The instability is promoted by increasing the sensitivity or by approaching the Chapman–Jouguet condition ($f = 1$).

For fixed values of the sensitivity and of the overdrive factor, the instability is promoted by increasing the stiffness of $\overline{w}'_N(\xi)$. At the bifurcation point, both the wavelength of the unstable disturbances and its oscillatory period decrease when the stiffness of $\overline{w}'_N(\xi)$ is increased. In the limit of a singular function $\overline{w}'_N(\xi)$, when n and m go to infinity, n/m fixed, the linear dynamics becomes pathological, as is also the case for the strongly overdriven detonations studied by Clavin *et al.* (1997). Moreover, a pathological dynamics occurs at bifurcation when the singularity is present only for the induction zone, with the decrease of the reaction rate in the exothermic zone being smooth.

Although the analytical result (8.16) remains regular in the limit $f = 1$, that limit is excluded in principle because of simplifications in § 8.2. The limit $f = 1$ is included in (8.3) and (8.4) with the downstream boundary condition (8.5) replaced by $Z = 0$, since constant values of Z at infinity must be excluded. According to (8.6), $L_{b+} = s$ and $L_{b-} = 0$, a zero multiplier should be used in (8.5) for that case. This system of equation deserves further numerical study for different structure of the unperturbed detonation. Since no divergence appears in the equations, no drastic changes would be expected from such analysis. However, no definitive answer concerning $f = 1$ can be given without a special analysis.

Useful future steps in improving our understanding of detonation dynamics are to carry out an analytical stability analysis involving both the entropy–vorticity wave and the acoustic waves, as is the case in ordinary gaseous detonations, and finally to develop a weakly nonlinear analysis of cellular detonations, generalizing the bifurcation analysis of Clavin & Denet (2002) at strong overdrive.

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Appendix A. The unperturbed solution

Let us introduce the notations

$$P=\frac{\gamma+1}{2\gamma}\left(\frac{\overline{p}}{\overline{p}_u}-1\right) \quad \text{and} \quad V=\frac{\gamma+1}{2}\left(\frac{\overline{\rho}_u}{\overline{\rho}}-1\right),$$

whence the equations for conservation of mass, momentum and energy across the planar detonation may be written as

$$P=-M_u^2V, \tag{A 1}$$

$$PV+P+V=\varepsilon^2Y, \tag{A 2}$$

to give

$$V=-\frac{1}{2M_u^2}\left[\left(M_u^2-1\right)+\sqrt{\left(M_u^2-1\right)^2-4\varepsilon^2M_u^2Y}\right], \tag{A 3}$$

where $Y=0$ at the Neumann state and $Y=1$ in the burnt-gas state, and where, in the same notations as in § 4, the propagation Mach number may be expressed in terms of the scaled overdrive factor defined in (4.2),

$$M_u=\sqrt{1+\varepsilon^2 f}+\varepsilon\sqrt{f}. \tag{A 4}$$

Equation (A 3) then yields

$$V=-(\sqrt{f}+\sqrt{f-Y})\left(\frac{\varepsilon}{M_u}\right),$$

$$V_N=-2\sqrt{f}\left(\frac{\varepsilon}{M_u}\right), \quad V_b=-(\sqrt{f}+\sqrt{f-1})\left(\frac{\varepsilon}{M_u}\right), \tag{A 5}$$

and the local sound speed and Mach number are expressed as

$$\frac{\overline{a}}{\overline{a}_u}=(1+(2/(\gamma+1))V)\,\frac{M_u}{M}, \qquad \frac{M^2}{M_u^2}=\frac{1+(2/(\gamma+1))V}{1-(2\gamma/(\gamma+1))M_u^2V}. \tag{A 6}$$

The Mach number in the burnt gas may be written in the form

$$1-M_b^2=\frac{\sqrt{f-1}}{\sqrt{f}}\left(M_u^2-1\right)\frac{1}{1+(\gamma/(\gamma+1))\left(1+\sqrt{f-1}/\sqrt{f}\right)\left(M_u^2-1\right)}, \tag{A 7}$$

which describes also the Chapman–Jouguet condition ($f=1$, $M_b=1$). The expansion of the quantity $1-M_b^2$ in powers of ε/M_u is obtained when the expansion of M_u^2-1, $M_u^2-1=2(\varepsilon/M_u)\sqrt{f}(1+2\varepsilon\sqrt{f}+\cdots)$, is introduced into (A 7),

$$1-M_b^2=2\left(\frac{\varepsilon}{M_u}\right)\sqrt{f-1}\left[1+\left(\frac{\varepsilon}{M_u}\right)\mathbb{M}\right] \tag{A 8}$$

with

$$\mathbb{M}=\mathbb{M}_2+O\left(\frac{\varepsilon}{M_u}\right), \qquad \mathbb{M}_2=\frac{2}{\gamma+1}(\sqrt{f}-\gamma\sqrt{f-1}). \tag{A 9}$$

In the same way, the expansion of the sound speed in the burnt gas is

$$\frac{\overline{a}_b}{\overline{a}_u} = 1 + \frac{\gamma - 1}{\gamma + 1}\left(\frac{\varepsilon}{M_u}\right)(\sqrt{f} + \sqrt{f-1}) + \cdots \tag{A 10}$$

which shows that the variation of the sound speed is of the second order in the distinguished limit $\varepsilon \to 0$, $(\gamma - 1) = O(\varepsilon)$, $f = O(1)$. Note that the relative variations of pressure and velocity across the detonation are opposite to leading order,

$$\left(\frac{\overline{p}}{\overline{p}_u} - 1\right) = -(\sqrt{f} + \sqrt{f - Y}) + O(\epsilon^2) = -\left(\frac{\overline{u}}{\overline{u}_u} - 1\right) + O(\epsilon^2), \tag{A 11}$$

or, with the notations of (4.8),

$$\overline{\pi} + \overline{\mu} = 1 + \epsilon\sqrt{f} + O(\epsilon^2). \tag{A 12}$$

Appendix B. Jump conditions for the unperturbed detonation

The perturbed boundary condition for pressure at the burnt-gas state, with modification to the internal detonation structure neglected, is obtained when M_u in (A 1) and (A 3) is replaced by $M_u - (\partial A/\partial t)/\overline{a}_u$,

$$\delta V_b = \frac{\sqrt{f} + \sqrt{f-1}}{\sqrt{f}}\left[1 - \frac{\varepsilon}{M_u}\left(\frac{2f - 1 + 2\sqrt{f}\sqrt{f-1}}{\sqrt{f} + \sqrt{f-1}}\right)\right]\dot{\mathscr{A}} \tag{B 1}$$

$$\delta P_b = M_u^2(2V_b\dot{\mathscr{A}} - \delta V_b), \tag{B 2}$$

$$\frac{\delta p_b}{\gamma \overline{p}_b} = \frac{2}{\gamma + 1}\frac{\overline{p}_u}{\overline{p}_b}\delta P_b, \tag{B 3}$$

where the notation $\dot{\mathscr{A}} = (\partial A/\partial t)/\overline{u}_u$ has been introduced. This yields

$$\frac{\delta p_b}{\gamma \overline{p}_b} = \frac{2}{\gamma + 1}\left(\frac{\sqrt{f} + \sqrt{f-1}}{\sqrt{f-1}}\right)\mathbb{P}\frac{\partial A/\partial t}{\overline{a}_b}, \tag{B 4}$$

with

$$\mathbb{P} = \frac{M_u}{1 - (2\gamma/(\gamma+1))M_u^2 V_b}\left[-1 + \frac{1}{\sqrt{f} + \sqrt{f-1}}\left(\frac{\varepsilon}{M_u}\right)\right]\frac{\overline{a}_b}{\overline{a}_u}, \tag{B 5}$$

$$\mathbb{P} = -1 + \left(\frac{\varepsilon}{M_u}\right)\sqrt{f} + \left(\frac{\varepsilon}{M_u}\right)^2\mathbb{P}_2, \tag{B 6}$$

where, to leading order in the distinguished limit $\varepsilon \to 0$, $(\gamma - 1) = O(\varepsilon)$, $f = O(1)$,

$$\mathbb{P}_2 = \frac{f-1}{2}. \tag{B 7}$$

The perturbed boundary conditions of the flow velocity is obtained from the equation for conservation of mass, $\overline{\rho}_b\delta u_b = (\overline{\rho}_b - \overline{\rho}_u)(\partial A/\partial t) - \overline{u}_b\delta\rho_b$, and transverse momentum, $\delta v_b = (\overline{u}_u - \overline{u}_b)\partial A/\partial y$, which may be written as

$$\frac{\delta u_b}{\overline{u}_b} = \frac{2}{\gamma + 1}\left(\frac{1}{1 + (2/(\gamma+1))V_b}\right)(\delta V_b - V_b\dot{\mathscr{A}}), \tag{B 8}$$

$$\frac{\delta v_b}{\overline{u}_b} = -\frac{2}{\gamma + 1}\left(\frac{V_b}{1 + (2/(\gamma+1))V_b}\right)\frac{\partial A}{\partial y}, \tag{B 9}$$

yielding

$$\frac{\delta u_b}{\bar{a}_b} = \frac{2}{\gamma+1}\left(\frac{\sqrt{f}+\sqrt{f-1}}{\sqrt{f-1}}\right)\left[1-\left(\frac{\varepsilon}{M_u}\right)\sqrt{f}\right]\frac{\partial A/\partial t}{\bar{a}_b}, \tag{B 10}$$

$$M_b\frac{\delta v_b}{\bar{a}_b} = \frac{2}{\gamma+1}\left(\frac{\sqrt{f}+\sqrt{f-1}}{\sqrt{f-1}}\right)\left(\frac{\varepsilon}{M_u}\right)(\sqrt{f-1})\left[1+\left(\frac{\varepsilon}{M_u}\right)\mathbb{V}_2\right]\frac{\partial A}{\partial y}, \tag{B 11}$$

with

$$\left[1+\left(\frac{\varepsilon}{M_u}\right)\mathbb{V}_2\right] = \frac{1}{1+(2/(\gamma+1))V_b}M_b^2, \qquad \mathbb{V}_2 = \mathbb{M}_2 + O\left(\frac{\varepsilon}{M_u}\right), \tag{B 12}$$

where $\mathbb{M}_2$ is given in (A 9).

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